Static Buckling of Moderately Thick, Anisotropic, Laminated and Sandwich Cylindrical Shell Panels

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This paper deals with the development of the linearized equations governing the buckling behavior of moderately thick, anisotropic, laminated, and sandwich cylindrical shell panels faced with fiber-reinforced plastic under uniform shear and normal membrane forces. The formulation takes into account the transverse shear flexibility. Two approaches for the approximate solution of the elastic stability equations are developed. The first approach is an analytical approach of the Galerkin type and rests upon the principle of virtual work. The second is a finite-element displacement approach. Some numerical results are presented and compared with other results from the open literature.

I. Introduction

IN recent years, the interest of the aerospace industries has been directed toward the multilayered fiber-reinforced composite structures for their high modulus, high strength, and low weight. The capability to predict the buckling behavior of anisotropic, laminated, and sandwich cylindrical shell panels faced with fiber-reinforced plastic, when subjected to combined shear and normal membrane loads, is of prime interest to structural analysts.

A particular characteristic of these laminated plates, in contrast to homogeneous plates, is the possible coupling between the in-plane extension and out-of-plane bending. Such coupling can significantly affect the response of these plates.[1]

Considerable literature exists on the static buckling of thin plates subjected to in-plane loadings. Reference 2 and, more recently, Ref. 3 give a summary of the work given on the buckling of isotropic and orthotropic thin plates with various boundary conditions. The buckling behavior of symmetrically and antisymmetrically laminated composite thin plates has been investigated by many authors.[4-9] An exhaustive compendium of the numerical results available for this problem can be found, among others, in Refs. 1 and 10.

Improvements of the classical plate theory (CLT), based on Kirchhoff assumptions, have been proposed by many researchers. First order, Reissner-Mindlin plate theories (FSDT), including the effect of the transverse shear deformation on the buckling behavior of laminated composite plates, have been used by a great number of investigators.[14-18] In Refs. 15-18 higher-order theories (HSDT) have also been developed. Exact three-dimensional elasticity solutions are given in Refs. 13, 19, and 20.

References 21-26 approach the problem of the general elastic stability of flat sandwich panels with laminated face plates.

A great number of theoretical works that use CLT to investigate the buckling behavior of laminated composite cylindrical shell panels exist.[27-31] Complicating effects, such as those due to transverse shear deformation are investigated in Refs. 32 and 33 and those due to sandwich construction in Refs. 34-37.

This brief review emphasizes also that only in a relatively few cases can exact solutions for the bifurcation buckling loads of anisotropic, laminated, and sandwich cylindrical shell panels be found—even under a simplifying assumption such as CLT.

For most structural configurations, approximate procedures such as the Rayleigh-Ritz, Galerkin, finite-difference, finite element, finite strip, and other methods must be used.

In Refs. 5, 14, 26, and 37, the principle of minimum potential energy in conjunction with the Rayleigh-Ritz method is used for the buckling analysis of anisotropic, laminated,[4,14] and sandwich plates and shell panels.[37]

The Galerkin method has been employed in Refs. 6 and 31 to obtain approximate solutions to stability equations of anisotropic, laminated plates[6] and shell panels[31] under arbitrary combinations of axial load, in-plane shear loadings, and internal pressure.

In Ref. 13 a finite-difference scheme has been applied to the equations of the three-dimensional elasticity theory; while an energy-based, finite-difference method has been used in Ref. 38 for investigating the buckling behavior of laminated composite circular cylindrical panels under prescribed uniform end-displacement.

Displacement or mixed finite-element approaches have been developed in Refs. 39-43 to analyze both bifurcation buckling and nonlinear behavior of laminated plates and shell panels. Sandwich constructions are not considered, and only eight-node quadratic quadrilateral finite elements are formulated. The finite strip method is developed in Ref. 44.

In the present paper, the linearized elastic stability equations of moderately thin, arbitrarily laminated, anisotropic, and sandwich shell panels under combined membrane loadings are derived. To this end, use is made of the principle of virtual work in conjunction with the Euler method of the adjacent equilibrium configurations[45,46] and of the Sanders nonlinear strain-displacement relations for cylindrical shells.[47] The formulation takes into account also the transverse shear deformation effects and gives, as particular cases, both the stability equations of the CLT and FSDT for anisotropic, laminated, and sandwich plates[1,6,12-15,34] and shell panels.[31,32,37]

Two approaches for searching the approximate solution to the stability equations are developed and compared. The first approach is an analytical approach of the Galerkin type that rests upon the direct use of the principle of virtual work. In

Received Feb. 27, 1989; revision received Nov. 14, 1989.

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this way, the set of the approximating functions do not necessarily satisfy all the boundary conditions (geometrical and mechanical), as required by the classical Galerkin's method. The second approach is a finite-element displacement approach. The geometric stiffness matrices for triangular and quadrilateral isoparametric, multilayered, Mindlin plate elements have been developed. Some convergence studies of the two approaches and numerical investigations have been performed and presented herein. The results are compared with other numerical results from open literature.

II. Displacement Field

Let us consider a multilayered, circular, cylindrical panel of constant thickness h, length a, and arc b composed of a finite number NS of thin orthotropic layers of uniform thickness perfectly bonded together. The material properties and the thickness of each layer may be entirely different. The coordinate system is shown in Fig. 1; the reference surface Ω, having contour C and radius R, is assumed to be the middle surface ($z = 0$).

For the purpose of developing a theory for laminated shells which accounts for the transverse shear deformation effects, the following displacement field is assumed[48]

$$\mathbf{u} = u + z(\gamma_x - w_{,x}); \quad \mathbf{v} = (1 - z/R)v + z(\gamma_y - w_{,y}); \quad \mathbf{w} = w \tag{1}$$

where $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are the displacements in the x-, y-, and z-directions, respectively; u, v, and w are midsurface displacements in the same coordinate directions; γ_x and γ_y are the shear rotations in the (x,z) and (y,z) planes, respectively, of a line normal to the reference surface. Notice that in the assumed displacement field, the effect of the transverse shear deformations appears in explicit form. In this paper, a comma denotes partial differentiation.

III. Strain-Displacement Field

If attention is restricted to problems where the membrane displacements are small and the largest rotations are $w_{,x}$ and $w_{,y} + v/R$, then the product not containing these rotations can be neglected in the expressions of finite strains.

Following Refs. 47 and 48, we assume for the strains in a cylindrical shell panel the following relations

$$\mathbf{e}_{xx} = e_{xx} + zk_x; \qquad \mathbf{e}_{yy} = e_{yy} + zk_y; \qquad \mathbf{e}_{zz} = 0$$
$$\mathbf{e}_{xy} = e_{xy} + zk_{xy}; \qquad \mathbf{e}_{xz} = e_{xz}; \qquad \mathbf{e}_{yz} = e_{yz} \tag{2}$$

where (In the present paper, it is assumed that $1 - z/R \approx 1$ will be adopted in geometrical, strain-displacement, and stress resultants relations.)

$$e_{xx} = u_{,x} + (w_{,x})^2/2 \qquad k_x = (\gamma_x - w_{,x})_{,x}$$
$$e_{yy} = v_{,y} - w/R + (w_{,y} + v/R)^2/2 \qquad k_y = (\gamma_y - w_{,y} - v/R)_{,y}$$
$$e_{xy} = u_{,y} + v_{,x} + w_{,x}(w_{,y} + v/R) \qquad e_{xz} = \gamma_x$$
$$k_{xy} = \gamma_{x,y} + \gamma_{y,x} - 2w_{,xy} - v_{,x}/R \qquad e_{yz} = \gamma_y \tag{3}$$

are the strains and curvature changes in the reference surface. These relations will be used in the next section to formulate the stability equations via the Euler method of the adjacent equilibrium configurations.

IV. Stability Equations Formulation

The linearized equations governing the bifurcation buckling of the laminated shell panels subjected to combined uniform inplane normal and shear loads $\bar{N}_{xx}$, $\bar{N}_{yy}$, and $\bar{N}_{xy}$ will be formulated by using the Euler method of the adjacent equilibrium configurations.[45,46] In this approach, we assume the existence of at least one additional, distinct equilibrium configuration (the adjacent equilibrium configuration) in the

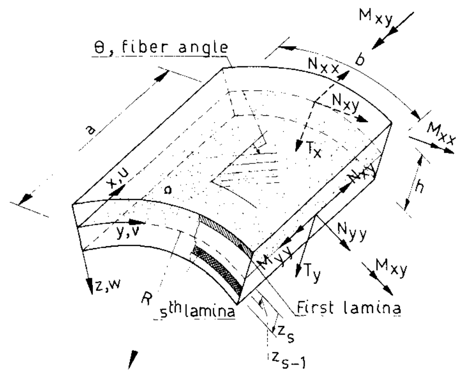


Fig. 1 Shell geometry and notations.

very close neighborhood of the original equilibrium configuration (the equilibrium configuration just prior to the occurrence of the buckling). If such an adjacent equilibrium configuration exists, the body may change suddenly from one equilibrium configuration (characterized by u^*, v^*, w^*, etc.) to the other under the stimulus of infinitesimal external disturbance (u^0, v^0, w^0, etc.)

Using the expressions for the nonlinear strains as given by Eqs. (3), on the assumption that the external forces and moments in both equilibrium configurations are identical, the principle of virtual work of the adjacent equilibrium configuration reads[45]

$$
\begin{aligned}
&\int_\Omega \{(N_{xx,x} + N_{xy,y})\delta u^0 + [(N_{xy} - M_{xy}/R)_{,x} \\
&+ (N_{yy} - M_{yy}/R)_{,y} - N^*_{xy} w^0_{,x}/R - N^*_{yy}(w^0_{,y} + v^0/R)/R \\
&- \underline{N_{xy} w^*_{,x}/R - N_{yy}(w^*_{,y} + v^*/R)/R}]\delta v^0 \\
&+ [M_{xx,xx} + 2M_{xy,xy} + M_{yy,yy} + N_{yy}/R \\
&+ (N^*_{xx} w^0_{,x} + N^*_{xy} w^0_{,y} + N^*_{xy} v^0/R)_{,x} \\
&+ (N^*_{xy} w^0_{,x} + N^*_{yy} w^0_{,y} + N^*_{yy} v^0/R)_{,y} \\
&+ \underline{(N_{xx} w^*_{,x} + N_{xy} w^*_{,y} + N_{xy} v^*/R)_{,x}} \\
&+ \underline{(N_{xy} w^*_{,x} + N_{yy} w^*_{,y} + N_{yy} v^*/R)_{,y}}]\delta w^0 \\
&+ (M_{xx,x} + M_{xy,y} - T_x)\delta\gamma^0_x + (M_{xy,x} + M_{yy,y} - T_y)\delta\gamma^0_y\} \, d\Omega \\
&- \int_\varrho \{(lN_{xx} + mN_{xy})\delta u^0 + [l(N_{xy} - M_{xy}/R) \\
&+ m(N_{yy} - M_{yy}/R)]\delta v^0 + [l(M_{xx,x} + M_{xy,y}) \\
&+ m(M_{xy,x} + M_{yy,y}) + l(N^*_{xx} w^0)_{,x} + m(N^*_{xy} w^0)_{,x} \\
&+ l(N^*_{xy} w^0)_{,y} + m(N^*_{yy} w^0)_{,y} + l(N^*_{xy} + mN^*_{yy})v^0/R \\
&+ \underline{l(N_{xx} w^*)_{,x} + m(N_{xy} w^*)_{,x} + l(N_{xy} w^*)_{,x} + m(N_{yy} w^*)_{,y}} \\
&+ \underline{(lN_{xy} + mN_{yy})v^*/R}]\delta w^0 + (lM_{xx} + mM_{xy})\delta(\gamma^0_x + w^0_{,x}) \\
&+ (lM_{xy} + mM_{yy})\delta(\gamma^0_y + w^0_{,y})\} \, dC
\end{aligned} \tag{4}
$$

where

$$(N_{xx}\; N_{yy}\; N_{xy}\; T_{xz}\; T_{yz}) = \langle(\sigma_{xx}\; \sigma_{yy}\; \sigma_{xy}\; \sigma_{xz}\; \sigma_{yz})\rangle \tag{5}$$

$$(M_{xx}\; M_{yy}\; M_{xy}) = \langle(\sigma_{xx}\; \sigma_{yy}\; \sigma_{xy})z\rangle \tag{6}$$

For notational convenience, here and below,

$$\langle\cdots\rangle = \sum_{s=1}^{NS} \int_{z_{s-1}}^{z_s} (\cdots)\, dz$$

As usual, the normal stress σ_{zz} is assumed to be small in comparison with other normal stresses and is neglected.

Notice that, according to the adjacent equilibrium criterion, terms of third or higher degree have been ignored in the above equation; in addition, all of the force and moment stress resultants which appear in Eq. (4) (with the exception of the starred forces) as well as the generalized displacements u^0, v^0, w^0, γ_x^0, and γ_y^0, are measured from the original equilibrium configuration, characterized by the starred values (that is, they represent the buckled mode). The incremental displacement field is of the form given by Eq. (1). Then the underlined terms in Eq. (4) represent the contributions of the prebuckling deformation state. In deriving Eq. (4) no assumptions on the (stress state) behavior of the panel before the onset of buckling have been made.

If before the onset of the buckling $w^* = \text{const}$, $v^* = 0$ (some of the conditions under which symmetrically and unsymmetrically laminated plates and shell panels under uniform biaxial compression and shear satisfy this assumption are discussed, for example, in Refs. 8, 49, and 50), the underlined terms give no contribution; in addition, the shell will be in a state of uniform stresses corresponding to the applied in-plane loads $\bar{N}_{xx}$, $\bar{N}_{yy}$, and $\bar{N}_{xy}$. In the following, it will be assumed that this assumption is verified, thus, reducing the stability problem to a classical linear (bifurcation) buckling problem.

Taking into account that the virtual variations are independent variations, the principle of virtual work yields the following set of equations governing the bifurcation buckling

$$\int_0^a \int_0^b \left(\sum_{j=1}^5 L_{ij} U_j \right) \delta U_i \, \mathrm{d}x \, \mathrm{d}y + B_i = 0 \qquad (i = 1,2,\ldots,5) \quad (7)$$

where the boundary terms B_i are

$$B_1 = -\int_0^b |N_{xx} \delta U_1|_0^a \, \mathrm{d}y - \int_0^a |N_{xy} \delta U_1| \, \mathrm{d}x$$

$$B_2 = -\int_0^b |(N_{xy} - M_{xy}/R) \delta U_2|_0^a \, \mathrm{d}y - \int_0^a |(N_{yy} - M_{yy}/R) \delta U_2|_0^b \, \mathrm{d}x$$

$$\begin{aligned} B_3 = & \int_0^b |[(M_{xx,x} + M_{xy,y} - \bar{N}_{xx} U_{3,x} - \bar{N}_{xy}(U_{3,y} + U_2/R)] \delta U_3|_0^a \, \mathrm{d}y \\ & + \int_0^a |[(M_{xy,x} + M_{yy,y} - \bar{N}_{xy} U_{3,x} - \bar{N}_{yy}(U_{3,y} + U_2/R)] \delta U_3|_0^b \, \mathrm{d}y \\ & - \int_0^b |M_{xx} \delta U_{3,x} + M_{xy} \delta U_{3,y}|_0^a \, \mathrm{d}y - \int_0^a |M_{xy} \delta U_{3,x} + M_{yy} \delta U_{3,y}|_0^b \, \mathrm{d}x \end{aligned}$$

$$B_4 = -\int_0^b |M_{xx} \delta U_4|_0^a \, \mathrm{d}y - \int_0^a |M_{xy} \delta U_4|_0^b \, \mathrm{d}x$$

$$B_5 = -\int_0^b |M_{xy} \delta U_5|_0^a \, \mathrm{d}y - \int_0^a |M_{yy} \delta U_5|_0^b \, \mathrm{d}x$$

For the elastic operators L_{ij} the following expressions hold

$$L_{11} = L_3^A; \quad L_{12} = L_4^A - L_4^B/R; \quad L_{13} = -L_{3,x}^B - L_{4,y}^B - L_1^A/R$$

$$L_{14} = L_3^B \quad L_{15} = L_4^B; \quad L_{22} = L_5^A - 2L_5^B/R + L_5^D/R^2 + \bar{N}_{yy}/R^2$$

$$L_{23} = -[(L_4^B - L_4^D/R)_{,x} + (L_5^B - L_5^D/R)_{,y} + (L_2^A - L_2^B/R)/R]$$

$$L_{24} = L_4^B - L_4^D/R; \qquad L_{25} = L_5^B - L_5^D/R$$

$$L_{33} = L_{3,xx}^D + 2L_{4,xy}^D + L_{5,yy}^D + 2(L_{1,x}^B + L_{2,y}^B)/R + A_{22}/R^2 + \bar{N}_{xx}(\cdot)_{,xx} + 2\bar{N}_{xy}(\cdot)_{,xy} + \bar{N}_{yy}(\cdot)_{,yy}$$

$$L_{34} = -L_{3,x}^D - L_{4,y}^D - L_1^B/R; \quad L_{35} = -L_{4,x}^D - L_{5,y}^D - L_2^B/R$$

$$L_{44} = L_3^D - A_{44}; \quad L_{45} = L_4^D - A_{45}; \quad L_{55} = L_5^D - A_{55}; \quad L_{ij} = L_{ji}$$

where

$$L_1^\tau = \tau_{12}(\cdot)_{,x} + \tau_{26}(\cdot)_{,y}; \quad L_3^\tau = \tau_{11}(\cdot)_{,xx} + 2\tau_{16}(\cdot)_{,xy} + \tau_{66}(\cdot)_{,yy}$$

$$L_2^\tau = \tau_{26}(\cdot)_{,x} + \tau_{22}(\cdot)_{,y}; \quad L_5^\tau = \tau_{66}(\cdot)_{,xx} + 2\tau_{26}(\cdot)_{,xy} + \tau_{22}(\cdot)_{,yy}$$

$$L_4^\tau = \tau_{16}(\cdot)_{,xx} + (\tau_{12} + \tau_{66})(\cdot)_{,xy} + \tau_{26}(\cdot)_{,yy}$$

and

$$U_1 = u^0; \quad U_2 = v^0; \quad U_3 = w^0; \quad U_4 = \gamma_x^0; \quad U_5 = \gamma_y^0$$

The differential operators which appear in the expressions for L_{ij} are obtained by replacing τ in L_i with the superscripts which appear in L_{ij}. These superscripts stand for the extensional, coupling, and bending stiffnesses whose expressions are given here

$$[A_{ij} \; B_{ij} \; D_{ij}] = \langle [1 \; z \; z^2] Q_{ij}(s) \rangle \qquad \text{for } i,j = 1,2,6$$

$$A_{ij} = X_{ij} \langle Q_{ij} \, \mathrm{d}z \rangle \qquad \text{for } i,j = 4,5 \qquad (8)$$

The expressions for L_{ij} follow from the constitutive equations

$$\{N\} = [A]\{\varepsilon^0\} + [B]\{k^0\}; \quad \{M\} = [B]\{\varepsilon^0\} + [D]\{k^0\};$$

$$\{T\} = [A^s]\{\gamma^0\}$$

where (T stands for transpose matrix)

$$\{N\}^T = (N_{xx} \; N_{yy} \; N_{xy}); \quad \{M\}^T = (M_{xx} \; M_{yy} \; M_{xy})$$

$$\{T\}^T = (T_{xy} T_{yz})$$

$$[A] = (A_{ij}); \quad [B] = (B_{ij}); \quad [D] = (D_{ij}); \quad [A^s] = (A_{ij})$$

for $i,j = 4,5$

$$\{\varepsilon^0\}^T = (\varepsilon_{xx}^0 \, \varepsilon_{yy}^0 \, \varepsilon_{xy}^0); \quad \{k^0\}^T = (k_x^0 \, k_y^0 \, k_{xy}^0); \quad \{\gamma^0\}^T = (\gamma_x^0 \, \gamma_y^0)$$

$$\varepsilon_{xx}^0 = u_{,x}^0 \qquad k_x^0 = (\gamma_x^0 - w_{,x}^0)_{,x}$$

$$\varepsilon_{yy}^0 = v_{,y}^0 - w^0/R \qquad k_y^0 = (\gamma_y^0 - w_{,y}^0 - v^0/R)_{,y}$$

$$\varepsilon_{xy}^0 = u_{,y}^0 + v_{,x}^0 \qquad k_{xy}^0 = \gamma_{x,y}^0 + \gamma_{y,x}^0 - 2w_{,xy}^0 - v_{,x}^0/R$$

In the above definitions, X_{ij} are the shear correction factors[51] and Q_{ij} the transformed lamina stiffnesses in the shell coordinate system (x,y,z).

In addition to the previous expression for A_{ij} $(i,j = 4,5)$, and for purposes of comparison of the numerical results with other results from the open literature, in the numerical calculations, we have also employed the following expressions:

for laminated homogeneous panels

$$A_{ij} = 5/4 \langle Q_{ij}[1 - (2z/h)^2] \rangle \qquad (9a)$$

for sandwich panels

$$A_{44} = (c + t) G_{xz}; \qquad A_{55} = (c + t) G_{yz} \qquad (9b)$$

where c and t are the thicknesses of core and faces, respectively, and G_{xz} and G_{yz} the transverse shear moduli of core in xz and yz planes, respectively.

Equation (9a) follows by assuming a parabolic distribution along the thickness of the transverse shear stresses and zero values at the bounding faces ($z = \pm h/2$).

From Eq. (4) or (7), the system of differential equations and boundary conditions governing the bifurcation buckling are readily derived:

Field equations

$$
\begin{aligned}
&N_{xx,x} + N_{xy,y} = 0 \\
&(N_{xy} - M_{xy}/R)_{,x} + (N_{yy} - M_{yy}/R)_{,y} \\
&\quad - [\bar{N}_{xy} w^0_{,x} - \bar{N}_{yy}(w^0_{,y} + v^0/R)]/R = 0 \\
&M_{xx,xx} + 2M_{xy,xy} + M_{yy,yy} + N_{yy}/R + \bar{N}_{xx} w^0_{,xx} \\
&\quad + \bar{N}_{xy}(w^o_{,y} + v^0/R)_{,x} + \bar{N}_{xy} w^0_{,xy} + \bar{N}_{yy}(w^0_{,y} + v^0/R)_{,y} = 0 \\
&M_{xx,x} + M_{xy,y} - T_x = 0 \\
&M_{xy,x} + M_{yy,y} - T_y = 0
\end{aligned}
\tag{10}
$$

or, in terms of generalized displacements,

$$
\sum_{j=1}^{5} L_{ij} U_j = 0 \qquad (i = 1,2,...,5) \tag{11}
$$

Boundary conditions

Natural Prescribed

along any side $x = \text{const}$

$$
\begin{aligned}
N_{xx} &= 0 & u^0 &= 0 \\
N_{xy} - M_{xy}/R &= 0 & v^0 &= 0 \\
M_{xx,x} + M_{xy,y} - \bar{N}_{xx} w^0_{,x} - \bar{N}_{xy}(w^0_{,y} + v^0/R) &= 0 & w^0 &= 0 \\
M_{xx} &= 0 & \gamma^0_x + w^0_{,x} &= 0 \\
M_{xy} &= 0 & \gamma^0_y + w^0_{,y} &= 0
\end{aligned}
\tag{12a}
$$

along any side $y = \text{const}$

$$
\begin{aligned}
N_{xy} &= 0 & u^0 &= 0 \\
N_{yy} - M_{yy}/R &= 0 & v^0 &= 0 \\
M_{xy,x} + M_{yy,y} - \bar{N}_{xy} w^0_{,x} - \bar{N}_{yy}(w^0_{,y} + v^0/R) &= 0 & w^0 &= 0 \\
M_{xy} &= 0 & \gamma^0_x + w^0_{,x} &= 0 \\
M_{yy} &= 0 & \gamma^0_y + w^0_{,y} &= 0
\end{aligned}
\tag{12b}
$$

As previously noted, Eqs. (7) or, alternatively, Eqs. (10) represent in some respect a generalization of the stability equations commonly used in the literature. First of all, Eqs. (7) and (8) can be considered the static counterpart of Eqs. (6–10) of Ref. 29 in which the transverse shear deformation effects have been taken into account. Second, let us neglect the contribution of v^0/R in the expression of the curvature and nonlinear terms of strain (Donnell-Mushtari approximation) and the contribution of the transverse shear stress resultants in the second of the stability of Eqs. (10) (shallow shell assumption). Then, the resulting equations give, for example, the stability equations of Ref. 32 and, for $\gamma^0_x = \gamma^0_y = 0$, the stability equations of Refs. 28 and 31. Making the curvature zero, the stability equations of Ref. 11, for symmetric laminates, and those of Ref. 13 for arbitrarily laminated plates are obtained. Furthermore, for sandwich plates, Eqs. (10) are equivalent to those of Ref. 34, and for sandwich shell panels faced with fiber-reinforced plastic under in-plane shear, it can be proved (see Ref. 52) that Eqs. (7) are equivalent to those given in Ref. 37 when the following substitutions are accomplished

$$
\gamma^0_y - w^0_{,y} - v^0/R = \beta_\alpha; \qquad \gamma^0_x - w^0_{,x} = \beta_x \tag{13}
$$

V. Approximate Solution

The system of Eqs. (10–12) gives the stability equations and boundary conditions. Except for simple structural systems, it is not possible to find an exact solution; then, we look for approximate solutions.

A. Approximate Analytical Formulation

The approximate analytical solutions are here obtained by making direct use of the principle of virtual work [Eqs. (7)]. Let

$$
U_i = \sum_{r=1}^{R} \sum_{t=1}^{T} C_{i,rt} F_{i,rt}(x,y) \qquad i = 1,5 \tag{14}
$$

where $C_{i,rt}$ are arbitrary parameters to be determined, and $F_{i,rt}$ are a priori chosen functions satisfying all the prescribed (geometric) boundary conditions. Upon substitution of the virtual variations for δU_i as given by Eq. (14) in Eq. (7), taking into account that the virtual variations are arbitrary, the required system of linear algebraic equations is obtained. For example, the first of Eqs. (7) would read

$$
\begin{aligned}
&\sum_{j=1}^{5} \sum_{r=1}^{R} \sum_{t=1}^{T} L^{1j}_{rt,pq} C_{j,rt} \\
&+ \int_0^b [N_{xx}(0,y) F_{1,pq}(0,y) - N_{xx}(a,y) F_{1,pq}(a,y)] \, \mathrm{d}y \\
&+ \int_0^a [N_{xy}(x,0) F_{1,pq}(x,0) - N_{xy}(x,b) F_{1,pq}(x,b)] \, \mathrm{d}x = 0
\end{aligned}
\tag{15}
$$

where $p = 1,R$ and $q = 1,T$ and, in general,

$$
L^{ij}_{rt,pq} = \int_0^a \int_0^b (L_{ij} F_{j,rt}) F_{i,pq} \, \mathrm{d}x \, \mathrm{d}y
$$

B. Finite-Element Formulation

As known, the accuracy of the finite-element solution depends on the ability of the assumed shape functions to accurately model the deformation modes of the structures. Since the coupling among extension, bending, and transverse shear deformation states affects the static, dynamic, and buckling behavior of the laminated anisotropic plates, a correct finite-element displacement formulation should model such a coupling.

The first author has previously developed[53] the small displacements stiffness matrices for several multilayered finite-plate elements using the displacement approach. Here we have focused our attention on the triangular and quadrilateral isoparametric elements. Sketches in Fig. 2 give the salient features of these elements. For further details, see Ref. 53. It suffices here to note that in each element the same set of shape functions is used for approximating all the nodal degrees of freedom, that is, the three displacement components u^0, v^0, and w^0 and the two rotation components $\beta^0_x = \gamma^0_x - w^0_{,x}$ and $\beta^0_y = \gamma^0_y - w^0_{,y}$. Here below we shall describe the main ideas underlying the development of the geometric stiffness matrices for finite-plate elements employed in the buckling analysis. Let

$$
\Phi = \tfrac{1}{2} \int_\Omega \{S^w\}^T [\bar{N}] \{S^w\} \, \mathrm{d}\Omega \tag{16}
$$

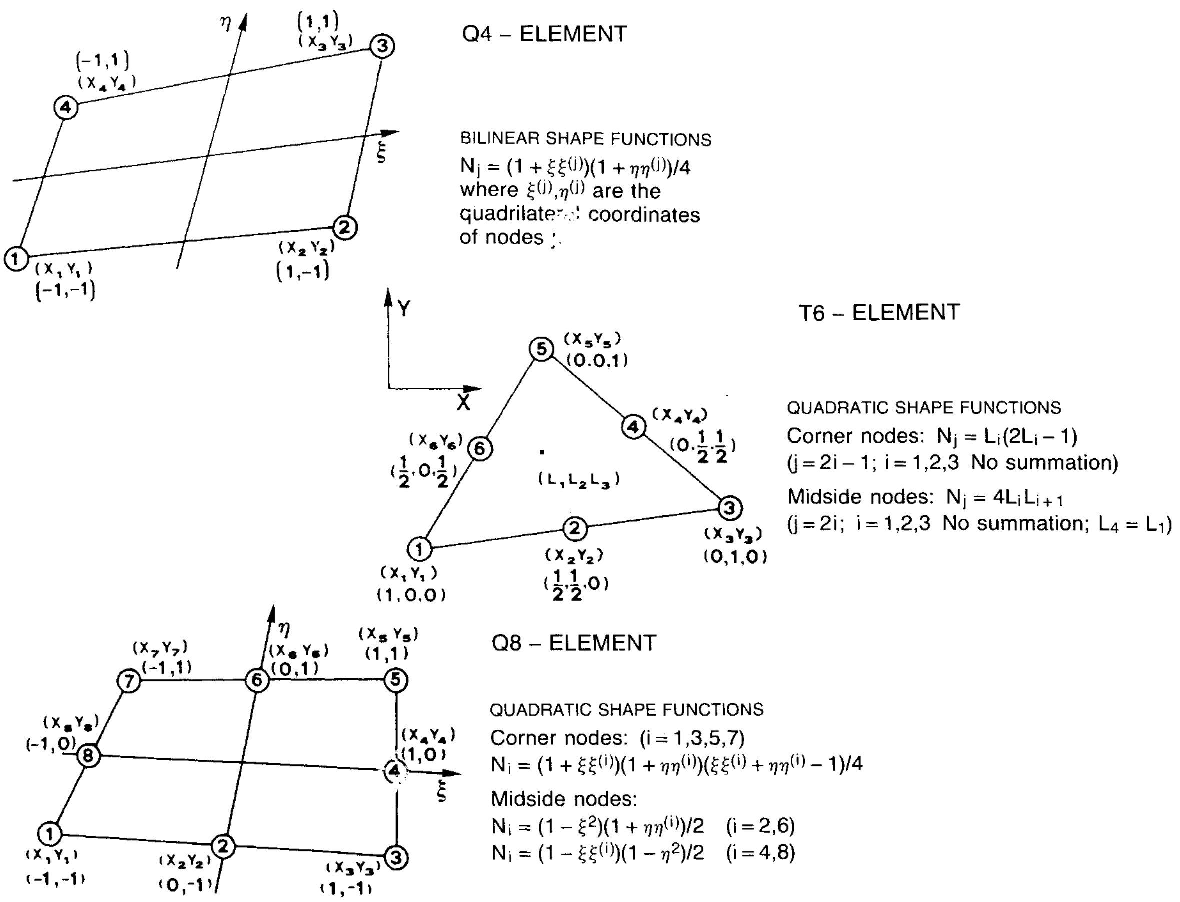


Fig. 2 Salient features of the finite elements investigated in the present analysis.

be the strain energy due to prebuckling stress state, where

$$\{S^w\} = \begin{Bmatrix} w^0_{,x} \\ w^0_{,y} \end{Bmatrix}; \qquad [\bar{N}] = \begin{bmatrix} \bar{N}_{xx} & \bar{N}_{xy} \\ \bar{N}_{xy} & \bar{N}_{yy} \end{bmatrix} \tag{17}$$

Let

$$\{\eta^w\}^T_e = [w^0_1 \cdots w^0_n]_e$$

be the vector of the degrees of freedom surrounding element e, relative to the transverse displacement with n as the number of the nodes of the element under consideration. With reference to Fig. 2,

$$w^0_e = \{N\}^T_e \{\eta^w\}_e \tag{18}$$

where $\{N\}_e$ is the appropriate matrix of the chosen shape functions and w_e refers to the transverse displacement of the points belonging to element e. Upon substitution of Eq. (18) into Eq. (16), the following expression for the geometric stiffness matrix is obtained

$$[K^w_\sigma]_e = \int_{\Omega^e} [\{N_{,x}\}_e \ \{N_{,y}\}_e] [\bar{N}] \begin{bmatrix} \{N_{,x}\}^T_e \\ \{N_{,y}\}^T_e \end{bmatrix} \mathrm{d}\Omega \tag{19}$$

Notice that the geometric stiffness matrix refers only to the transverse displacement w^0. Let $[K_\sigma]_e$ be the overall geometric stiffness matrix of element e, and $\{\eta\}_e$ be the vector of degrees of freedom surrounding element e,

$$\{\eta\}^T_e = [\{\eta^u\}^T_e \ \{\eta^w\}^T_e \ \{\eta^\beta\}^T_e]$$

with

$$\{\eta^u\}^T_e = [u^0_1 \cdots u^0_n \ v^0_1 \cdots v^0_n]$$

$$\{\eta^\beta\}^T_e = [\beta^0_{x1} \cdots \beta^0_{xn} \ \beta^0_{y1} \cdots \beta^0_{yn}]$$

Then, in partitioned form

$$[K_\sigma]_e = \begin{bmatrix} [0] & [0] & [0] \\ [0] & [K^w_\sigma]_e & [0] \\ [0] & [0] & [0] \end{bmatrix}$$

VI. Numerical Results and Discussion

In order to assess the accuracy and reliability of the proposed approaches, some numerical investigation has been performed.

To the best of the author's knowledge, there exist only two sets of functions which lead to exact solutions for laminated plates and shells subjected to biaxial compression. The first set, namely ($\alpha_r = r\pi x/a$; $\beta_t = t\pi y/b$)

$$F_{1,rt} = F_{4,rt} = \cos\alpha_r \sin\beta_t; \quad F_{2,rt} = F_{5,rt} = \sin\alpha_r \cos\beta_t$$

$$F_{3,rt} = \sin\alpha_r \sin\beta_t \tag{20}$$

satisfies the following simple support boundary conditions (hinged edges, free in the in-plane normal direction[54])

$$x = 0,a: \qquad N_{xx} = M_{xx} = U_2 = U_3 = U_5 = 0$$

$$y = 0,b: \qquad N_{yy} = M_{yy} = U_1 = U_3 = U_4 = 0 \tag{21}$$

for cross-ply laminates and leads to an exact solution for symmetric and antisymmetric cross-ply laminates. The second set, namely

$$F_{1,rt} = F_{5,rt} = \sin\alpha_r \cos\beta_t; \qquad F_{2,rt} = F_{4,rt} = \cos\alpha_r \sin\beta_t$$

$$F_{3,rt} = \sin\alpha_r \sin\beta_t \tag{22}$$

Table 1 Convergence study for the analytical solution, symmetric cross-ply and angle-ply square plates ($a/h = 10$), buckling load parameter $\lambda = N_{cr}a^2/E_T h^3$

$R = T$	2	4	6	8	Exact solution
*	(35.83)	(35.83)	(35.83)	(35.83)	(35.83)
	24.17	24.17	24.17	24.17	(24.17)
**	(±99.45)	(±85.04)	(±84.49)	(±84.42)	——
	±48.91	±33.65	±33.52	±33.22	
***	(17.21)	(17.17)	(17.17)	(17.17)	——
	11.01	10.86	10.84	10.84	
			Layup (0/90/0/90/0 deg)		
*	(58.61)	(53.83)	(52.36)	(51.58)	——
	32.59	30.09	29.68	29.50	
**	(+127.15)	(+63.08)	(+60.99)	(+60.11)	——
	+60.43	+30.43	+29.82	+29.56	
	(−247.85)	(−229.76)	(−229.70)	(−229.30)	
	−78.66	−46.26	−44.76	−44.62	
***	(25.15)	(20.84)	(20.14)	(19.81)	——
	14.70	12.18	11.89	11.76	
			Layup (45/−45/45 deg)		

Table 2 Convergence study for the analytical solution, antisymmetric cross-ply and angle-ply square plates ($a/h = 10$), buckling load parameter $\lambda = N_{cr}a^2/E_T h^3$

$R = T$	2	4	6	8	Exact solution
*	(12.63)	(12.63)	(12.63)	(12.63)	(12.63)
	11.10	11.10	11.10	11.10	11.10
**	(±35.05)	(±30.57)	(±30.38)	(±30.36)	——
	±26.38	±20.82	±20.58	±20.52	
***	(6.07)	(6.05)	(6.05)	(6.05)	——
	5.23	5.20	5.19	5.19	
			Layup (0/90 deg)		
*	(58.22)	(51.40)	(49.36)	(48.33)	——
	32.28	28.93	28.26	27.95	
**	(±161.60)	(±73.16)	(±70.01)	(±68.54)	——
	±67.17	±32.43	±31.51	±31.17	
***	(27.96)	(23.04)	(22.07)	(21.59)	——
	15.18	13.10	12.75	12.58	
			Layup (45/−45 deg)		

Table 3 Convergence study for the analytical solution, symmetric cross-ply and angle-ply square shell panels ($R/h = 100$, $a/h = 10$), buckling load parameter $\lambda = N_{cr}a^2/E_T h^3$

$R = T$	2	4	6	8	Exact solution
*	(35.88)	(35.88)	(35.88)	(35.88)	(35.88)
	24.22	24.22	24.22	24.22	24.22
**	(±99.53)	(±85.08)	(±84.54)	(±84.46)	——
	±48.96	±33.57	±33.34	±33.24	
***	(17.23)	(17.19)	(17.19)	(17.19)	——
	11.03	10.88	10.86	10.85	
			Layup (0/90/0/90/0 deg)		
Present	−460.62	−103.65	−89.90	−89.22	——
**	500.89	201.97	201.82	201.68	
Ref. 5	−475.59	−105.64	−92.27	−91.39	——
**	518.36	208.45	207.25	207.18	
		Layup (45/−45/45 deg); geometry and material as Ref. 5, only CLT values			

Table 4 Convergence study for the analytical solution, antisymmetric cross-ply and angle-ply square shell panels ($R/h = 100$, $a/h = 10$), buckling load parameter $\lambda = N_{cr}a^2/E_Th^3$

$R = T$	2	4	6	8	Exact solution
*	(12.68)	(12.68)	(12.68)	(12.68)	(12.68)
	11.15	11.15	11.15	11.15	11.15
**	(±35.13)	(±30.62)	(±30.43)	(±30.41)	——
	±26.45	±20.86	±20.62	±20.56	
***	(6.09)	(6.08)	(6.08)	(6.08)	——
	5.25	5.22	5.22	5.22	
			Layup (0/90 deg)		
*	(58.71)	(51.85)	(49.79)	(48.75)	——
	32.35	29.01	28.33	28.02	
	(+159.52)	(+72.29)	(+69.10)	(+67.62)	
**	(−165.21)	(−74.45)	(−72.31)	(−69.86)	——
	+67.19	+32.31	+31.39	+31.03	
	−68.21	−32.52	−31.62	−31.34	
***	(27.98)	(22.89)	(21.89)	(21.89)	——
	15.97	13.06	12.70	12.70	
			Layup (45/−45 deg)		

leads to an exact solution for antisymmetric angle-ply laminates having the following simple support boundary conditions (hinged edges, free in the in-plane tangential direction[54])

$$x = 0,a: \quad N_{xy} = M_{xx} = U_1 = U_3 = U_5 = 0$$

$$y = 0,b: \quad N_{xy} = M_{yy} = U_2 = U_3 = U_4 = 0 \tag{23}$$

In this work, following Whitney,[31] we considered the boundary conditions of Eqs. (21) and select Eqs. (20) for expressing the unknown functions in the approximate analytical solution.

The numerical results refer to the following mechanical and geometrical properties:

1) Laminated composite panels

$$E_L/E_T = 40; \; G_{LT}/E_T = 0.5; \; G_{TT}/E_T = 0.6; \; \nu_{LT} = \nu_{TT} = 0.25$$

2) Sandwich panels with laminated faces (unit N/mm²) type (a) $a = b = 225$ mm; faces

$$E_L = 229000; \quad E_T = 13350; \quad G_{LT} = 5249; \quad \nu_{LT} = 0.3151$$

$t = 0.225$ mm; layup (45/−45/0/45/−45 deg) thicknesses (0.025/0.125/0.025) core

$$E_L = E_T = 828; \quad G_{xz} = 146; \quad G_{yz} = 90.4; \quad c = 4.775 \text{ mm}$$

type (b) $a = b = 225$ mm; faces

$$E_L = 24210; \quad E_T = 5433; \quad G_{LT} = 2452; \quad \nu_{LT} = 0.334$$

$t = 0.315$ mm; layup (45 deg) core

$$G_{xz} = 10.5; \qquad G_{yz} = 21; \qquad c = 6.615 \text{ mm}$$

and loading conditions: * uniform axial compression (x-direction); ** in-plane shear; *** uniform biaxial compression and in-plane shear. Here E_L and E_T are major and minor Young's moduli in the principal material directions (L,T), ν_{LT} and ν_{TT} are Poisson's ratios, and G_{LT} and G_{TT} are the shear moduli. The other quantities have been previously defined.

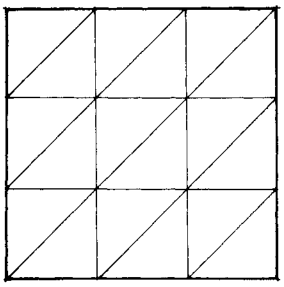
With the exception of the symmetric angle-ply laminate 45/−45/45 deg investigated in Table 3 for the purpose of comparison with the results obtained in Ref. 31, all of the laminates are regular laminates, that is, all the constituent layers have the same thickness. The three layers in this symmetric angle-ply laminate have thicknesses ($h/4$;$h/2$;$h/4$).

All the numerical results quoted in the present work were obtained by taking into account the FSDT with a shear correction factor $X = 5/6$. To give some insight into the approximation obtainable using the CLT, values based on the CLT are given enclosed in brackets.

In Table 1 the convergence characteristics of the approximate analytical solution is shown for symmetric cross-ply and angle-ply laminates. As mentioned above, for the symmetric and antisymmetric cross-ply laminates subjected to uniform axial compression, the assumed approximate solution [Eqs. (20)] leads to an exact solution. In Table 2 the convergence analysis is performed on two different antisymmetric (cross-ply and angle-ply) sequences. As previously noted, Eqs. (22) lead to an exact solution for antisymmetric angle-ply laminates subjected to boundary conditions [Eq. (23)]. Since in this paper we considered boundary conditions [Eq. (21)], it follows that the antisymmetric angle ply has no exact solution even for the simple case of uniform axial compression.

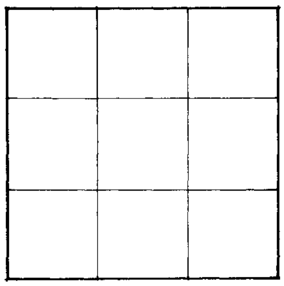
Notice that for all of the investigated stacking sequences, the rate of convergence decreases for the case of pure in-plane shear loading and angle-ply laminates.

Table 3 gives the results of the convergence analysis for the analytical solution of the buckling load parameter of symmetric cross-ply and angle-ply square shell panels; while Table 4 gives the results for antisymmetric cross-ply and angle-ply



a)

3X3 MESH FOR TRIANGULAR ELEMENTS



b)

3X3 MESH FOR QUADRILATERAL ELEMENTS

Fig. 3 Denomination of the finite element mesh.

square shell panels. The remarks for Tables 1 and 2 appear to hold.

Table 5 shows the convergence characteristics of the developed finite elements; Fig. 3 gives the finite-element meshes employed in the present study. The numerical results quoted in this table give rise to some interesting considerations. First, it appears that finite-element results decrease monotonically for increasing number of elements in the mesh and give upper bounds for the exact solution, where these exist. The $T6$ and $Q8$ finite elements appear to give converged results for the 4×4 and 5×5 meshes. If this is assumed to be true also for the layups and boundary conditions for which no exact solutions exist, then it could be concluded that Galerkin's approximate solutions obtained with $R = T = 8$ are not converged, especially for the layup $45/-45$ deg with boundary conditions as given by Eq. (21). Given the evident superiority of the $T6$ and $Q8$ finite elements, in the following, only the $Q8$ finite element will be employed further. Convergence characteristics of the $Q8$ finite element for four different stacking sequences (regular antisymmetric and symmetric cross-ply and angle-ply laminates) and three different loading conditions are shown in Table 6.

As in the approximate analytical solution, the lowest rate of convergence appears to be for laminates under pure in-plane shear loading or for angle-ply laminates under any one of the loading conditions. It is very interesting to notice that the discrepancy between finite element and Galerkin's approximate solutions for the $45/-45$ deg layup under uniform axial compression; it is not so marked for the other two loading conditions. Beyond this, the same conclusions as those given for the results in Table 5 appear to hold.

In Figs. 4 and 5, the behavior of the buckling load parameter for symmetric cross-ply (0/90/0/90/0 deg) and angle-ply ($45/-45/45$ deg) square plates as a function of the plate thickness ratio a/h is plotted. Notice that at moderately low values of the plate thickness ratio a/h, the solution given by the CLT is not conservative. Then, the CLT is inadequate to predict the buckling behavior of moderately thick laminates or thin laminates which exhibit high anisotropy ratios. It should also be noticed that for angle-ply the differences between the CLT and FSDT previsions are greater than for cross-ply laminates (Tables 1–4), and, for the same stacking sequence, for shear in-plane load than for axial compression (Tables 1–4), and for negative shear than for positive shear (Table 1).

To give some insight on the effect of the curvature, in Table 7, results are given on the behavior of the buckling load parameters for two square shell panels having the same lamination schemes as those in Figs. 4 and 5. Notice that for the assumed value of the curvature parameter, there is no appreciable difference from the results for the corresponding plates (Figs. 4 and 5); however, due to curvature, the CLT solution is also a monotone increasing function of the ratio a/h.

In Fig. 6 the behavior of the buckling load parameters for symmetric cross-ply plates under uniaxial compression or

Table 5 Convergence study of the finite-element solution for buckling load parameter $\lambda = N_{xx}a^2/E_T h^3$, square plate ($a/h = 10$), numbers in [···] give the unconstrained degree of freedom; A.S. = Analytical approximate solution; E.S. = Exact solution.

	Mesh								
	3×3	4×4	5×5	3×3	4×4	5×5	3×3	4×4	5×5
$T3$	44.08 [36]	35.79 [69]	31.79 [128]	32.83	23.57	19.23	——	——	——
$Q4$	32.23 [36]	28.60 [69]	26.98 [128]	19.95	15.92	14.14	25.52	21.93	20.32
$T6$	24.92 [165]	24.43 [301]	—— ——	11.89	11.38	——	17.96	17.80	——
$Q8$	24.32 [120]	24.22 [221]	24.19 [262]	11.24	11.15	11.12	17.67	17.59	17.41
Layup		(0/90/0/90/0 deg)			(0/90 deg)			$(45/-45\ \text{deg})^a$	
A.S. ($R = T = 8$)		24.17			11.10			17.55	
E.S.		24.17			11.10			17.55	
$T3$	53.45 [36]	44.76 [69]	39.64 [128]	47.17	36.17	30.53			
$Q4$	41.46 [36]	36.09 [69]	33.05 [128]	37.70	29.55	25.82			
$T6$	31.04 [165]	29.12 [301]	—— ——	21.85	20.41	——			
$Q8$	29.40 [120]	28.17 [221]	27.86 [262]	21.58	20.05	19.72			
Layup		($45/-45/45$ deg)			($45/-45$ deg)				
E.S.		29.50			27.95				

[a]Boundary conditions as given by Eq. (23).

Table 6 Convergence study of the $Q8$ finite element for different laminate configurations and loading conditions, square plate ($a/h = 10$), buckling load parameter $\lambda = N_{cr}a^2/E_T h^3$

Loading	Mesh			A.S. $R = T = 8$	Mesh			A.S. $R = T = 8$
	3×3	4×4	5×5	FSDT	3×3	4×4	5×5	FSDT
*	24.32	24.22	24.19	24.17	29.40	28.17	27.86	29.50
**	± 41.68	± 35.47	± 34.07	± 33.22	$+34.64$	$+31.25$	$+29.75$	$+29.56$
					-51.08	-47.06	-46.01	-44.62
***	11.14	10.92	10.86	10.84	12.32	11.55	11.33	11.76
	Layup (0/90/0/90/0 deg)				Layup ($45/-45/45$ deg)			
*	11.24	11.15	11.12	11.10	21.58	20.05	19.72	27.95
**	± 24.25	± 21.40	± 20.79	± 20.52	± 34.01	± 30.78	± 29.52	± 31.17
***	5.35	5.24	5.21	5.19	10.88	10.41	10.28	12.58
	Layup (0/90 deg)				Layup ($45/-45$ deg)			

shear is given as a function of the aspect ratio a/b. As expected, CLT overestimates buckling loads for all the values of the aspect ratio, the error is greater for $a/b < 1$ because in this range, $a/h < b/h$; in addition, also the critical mode shapes are not predicted correctly.

The following results refer to sandwich panels with laminated faces. Table 8 shows the results of the convergence analysis of the $Q8$ finite element. With the exception of the in-plane shear condition, the 5×5 mesh appears to give fairly well converged results. For this last loading condition, the results appear to be not converged enough and do not agree very well with the Galerkin's previsions, particularly for the negative in-plane shear buckling load of sandwich plate type (b). Table 9 gives a comparison of the results obtained in the present work with others quoted in the literature. To illustrate the influence of the transverse shear stiffness coefficient values, the calculations have been performed using for A_{44}, A_{45}, and A_{55}, the two expressions given by Eqs. (8) and (9). In Ref. 34 the transverse shear stiffnesses have been calculated from Eq. (9b); while in Ref. 37 the faces contribution has been neglected, that is, it has been assumed that $c + t = c$ in Eq. (9b). Notice that the shear buckling load is sensitive much greater than the axial buckling load to this variation. The results obtained in the present study agree very well with those

Table 7 Effect of the length-to-thickness ratio a/h on the buckling load parameter $\lambda = N_{cr}a^2/E_T h^3$, symmetric cross-ply and angle-ply square shell panels ($R/a = 20$), $R = T = 6$

	Length-to-thickness ratio, a/h				
Loading	10	20	30	50	100
*	24.19	31.91	34.04	35.42	36.86
	(35.84)	(35.88)	(35.94)	(36.13)	(37.04)
**	±33.32	±60.56	±71.72	±79.56	±84.23
	(±84.51)	(±84.54)	(±84.59)	(±84.78)	(±85.62)
***	10.85	14.99	16.16	16.91	17.61
	(17.17)	(17.19)	(17.22)	(17.31)	(17.72)
	Layup (0/90/0/90/0 deg)				
*	29.70	45.88	50.06	54.04	63.53
	(52.48)	(52.84)	(53.43)	(55.32)	(64.01)
	29.84	48.67	55.19	59.97	65.41
	−44.77	−119.04	−163.98	−203.27	−233.66
**	(61.05)	(61.23)	(61.53)	(62.46)	(66.14)
	(−229.82)	(−230.22)	(−230.89)	(−233.00)	(−242.23)
***	11.92	17.31	19.03	20.54	23.65
	(20.18)	(20.29)	(20.49)	(21.10)	(23.81)
	Layup (45/−45/45 deg)				

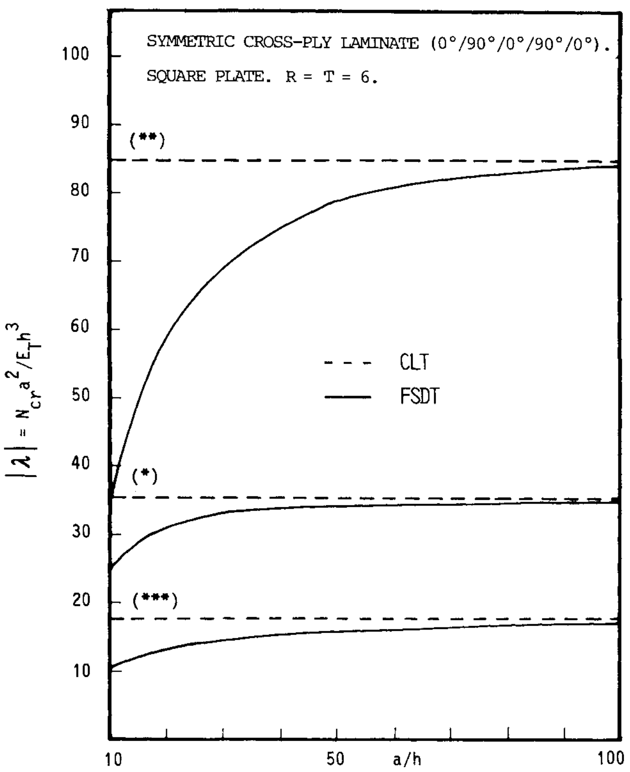


Fig. 4 Comparison between CLT and FSDT theories for buckling under axial, in-plane shear, and combined loads.

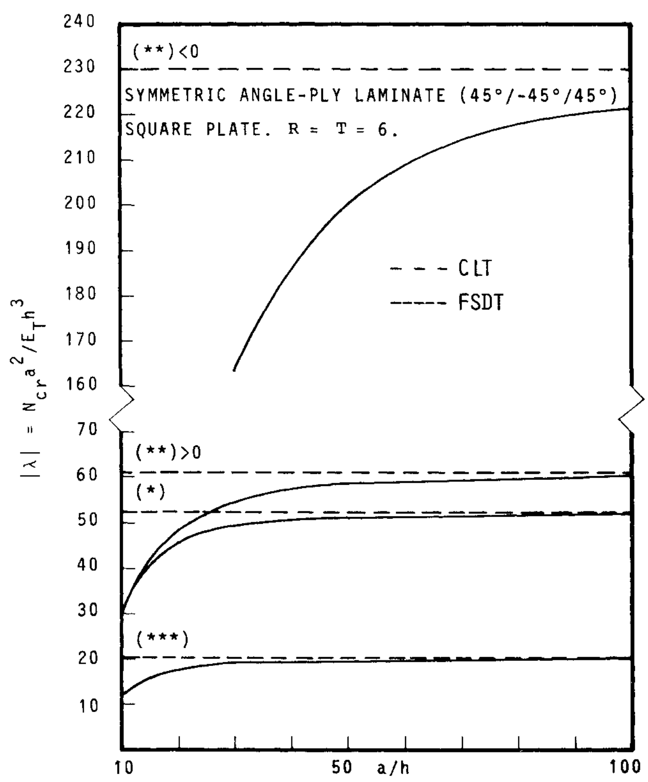


Fig. 5 Comparison between CLT and FSDT theories for buckling under axial, in-plane shear, and combined loads.

obtained by the other researchers. In Fig. 7 the buckling loads are plotted as a function of the panel curvature. Some considerations are to be made about these results. First of all, as expected, an increase in the curvature produces an increase in the buckling load, thereby, countering the effects of transverse shear deformability which tends to reduce the buckling load (see Figs. 4 and 5). This beneficial effect of the curvature is greater for the axial compression than for the in-plane shear buckling. In addition, it should be noticed that for small values of R, the negative shear stability load is smaller than the positive. The reverse is true for high values of R.

In Fig. 8, the behavior of the buckling load parameter vs the ratio c/t (thickness core/thickness face) for a square sandwich plate of the type (b) with $a/h = 10$ and $G_{xz} = G_{yz} = 10.5$ is plotted. For the investigated structural configuration, it appears that the buckling load for uniaxial compression is more sensitive than the buckling loads for the other two loading conditions to the variation in the values of this parameter. A common feature displayed by these plots is the trend to increase in the buckling load of increasing value of c/t, in the range of the low values of this parameter. The reverse is true in the range of the higher values. Notice that the plate is more stable under negative than positive shear and that the value of the negative shear buckling load is almost independent of the value c/t.

In Fig. 9, for the sandwich plate investigated in Fig. 8, the effect of the core stiffness G_{xz} on the buckling loads is shown. The variation of the ratio E_T/G_{xz} (minor Young's modulus of face/transverse shear modulus of core) is here obtained by varying the value of the G_{xz} and taking constant that of E_T. An increase in each of the buckling loads is observed as the core stiffness increases. In addition, at values of E_T/G_{xz} higher than 100, the positive and negative shear buckling loads are almost equal.

Table 8 Convergence study of the $Q8$ finite element for sandwich plates, buckling load N/mm, A_{ij} $(i,j = 4,5)$ as given by Eq. (9b)

Loading	Mesh 3 × 3	Mesh 4 × 4	Mesh 5 × 5	A.S. $R = T = 8$	Exact solution
*	9.94	9.69	9.46	9.37	9.37
**	±18.49	±16.20	±14.71	±13.50	——
***	7.57	7.03	6.67	6.24	——
	Sandwich plate type (b) with faces (0/90/0/90/0 deg)				
*	49.63	48.53	48.21	47.69	——
**	+69.31	+56.49	+49.73	+50.89	——
	−99.87	−89.53	−81.76	−85.25	
***	20.26	19.09	18.77	18.62	——
	Sandwich plate type (b)				

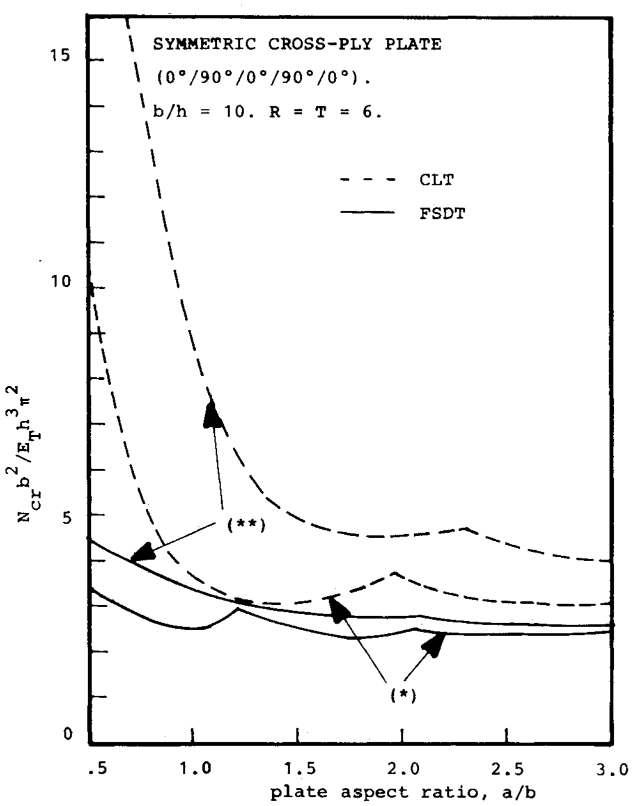


Fig. 6 Effect of the aspect ratio on the buckling load parameter.

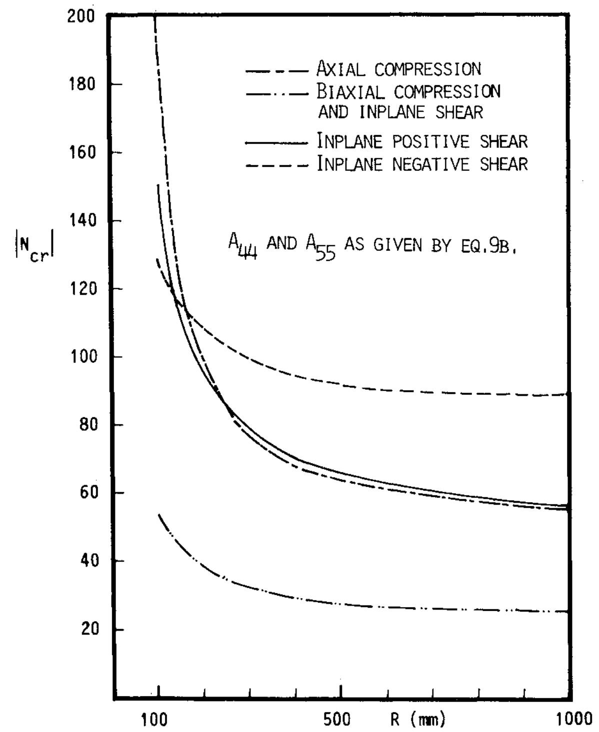


Fig. 7 Effect of curvature on the buckling load (N/mm) of sandwich panel type (b).

Table 9 Results of comparison with Refs. 34 and 37, sandwich plates, buckling load N/mm

Loading	Present $R = T = 4$ Eq. (9a)	Eq. (9b)			Sandwich
*	174.40	167.70	172.50	(Ref. 34)	Type a
	68.09	52.28	53.86		
**				(Ref. 37)	Type b
	−140.00	−86.48	−86.08		

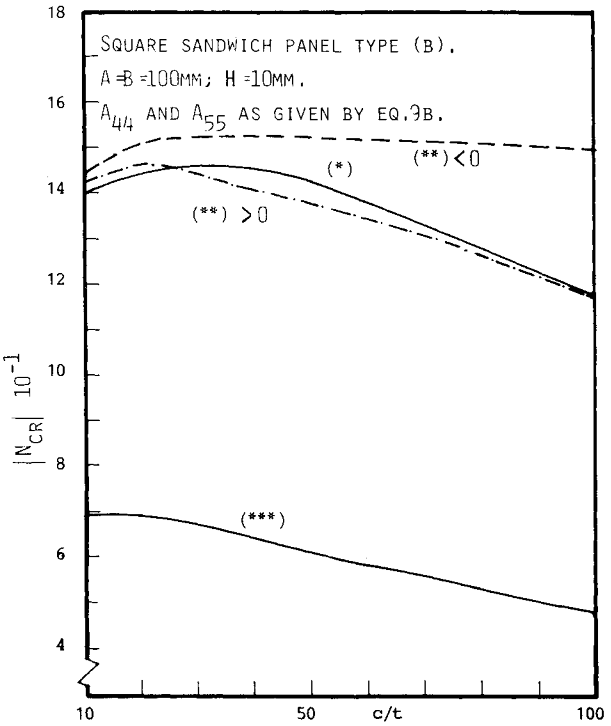


Fig. 8 Variation of the buckling load (N/mm) with c/t.

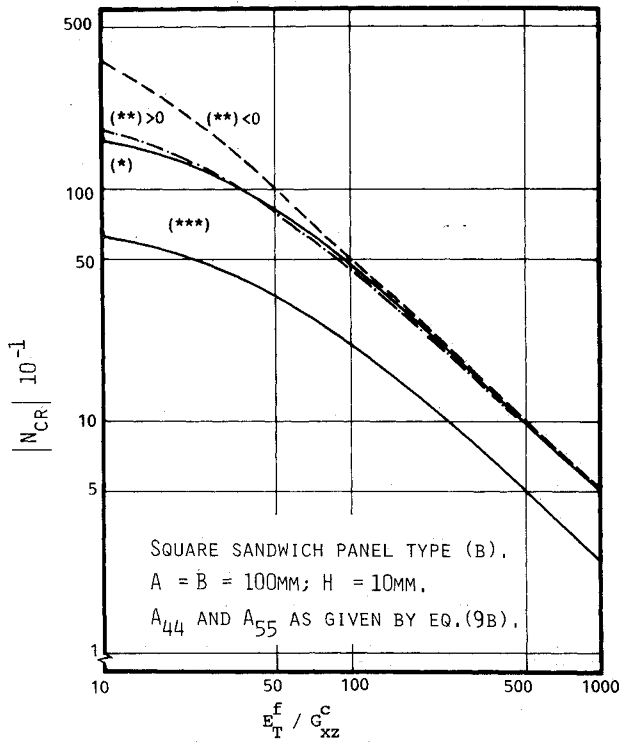


Fig. 9 Variation of the buckling load (N/mm) with E^f_T/G^c_{xz}.

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